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THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. VII. MARCH, 1900. No. 3.

ON THE TRANSCENDENTAL FORM OF THE RESULTANT.

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(A paper presented December 28, 1899, at the meeting of the Chicago Section of the American Mathematical Society.)*

I. DERIVATION AND PROPERTIES.

§ 1. INTRODUCTION.

The writer was incited to the following investigation by portions of letters written by Professor Gordan to M. Hermite and himself. Under date of October 17, 1898, in a letter to the writer Professor Gordan says: "Ich habe die Untersuchung der Resultante weiter verfolgt und die Formel gefunden, sie durch Potenzsummen s der beiden Gleichungen auszudruecken. Ich habe sie nach Paris gesandt, wo sie demnächst in den Comptes Rendus erscheinen wird. R sei die Resultante der Gleichungen

$$f=x^m+a_1x^{m-1}+a_2x^{m-2}+.\ \ldots.$$

$$1+\phi=1+b_1x+b_2x^2+\ldots\ldots$$

Die $s_\rho\,(a)$ sind die Potenzsummen der Wurzeln α der Gleichung f und die $s_\rho\,(b)$

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*The writer wishes here to acknowledge his obligations to Prof. H. S. White for presenting this paper and for subsequent criticisms for the revision. He wishes also to thank Prof. M. Bocher for friendly criticism.

dieselben Functionen der b. Die Producte der s bezeiche ich durch S: $S_\lambda = (s_{\lambda_1})^{\rho_1}(s_{\lambda_2})^{\rho_2} \ldots$ und fuehre die Zahl g_λ ein, $g_\lambda = (\lambda_1)^{\rho_1}(\lambda_2)^{\rho_2} \ldots . \rho_1 ! \rho_2 ! \ldots .$ Es ist dann :

$$R = \Sigma \frac{(-1)^{\rho_1 + \rho_2 + \ldots\ldots}}{g_\lambda} S_\lambda(a) \, S_\lambda(b).$$

Die Summe erstreckt sich ueber alle λ, ρ, fuer welche

$$\lambda_1 \rho_1 + \lambda_2 \rho_2 + \ldots\ldots \leqq mn$$

ist.

Aus diesser Formel kann man leicht die Entwickelung der symmetrischen Functionen

$$\Sigma \alpha_1^{\kappa_1} \alpha_2^{\kappa_2} \alpha_3^{\kappa_3} \ldots\ldots = \beta_{\kappa_1} \beta_{\kappa_2} \beta_{\kappa_3} \ldots\ldots$$

sowohl als ganze Function der s als auch als ganz Function der Coefficienten a ableiten. In der Formel

$$\beta_{\kappa_1} \beta_{\kappa_2} \ldots\ldots \beta_{\kappa_\nu} = \Sigma (-1)^{\nu + \rho_1 + \rho_2 + \ldots\ldots} C_{\lambda,\kappa} S_\lambda$$

sind die Coefficienten $C_{\lambda,\kappa}$ mittelst der Formel gegeben :

$$C_{\lambda,\kappa} = \Sigma \frac{1}{\tau_1 ! \tau_2 ! \ldots .} (\varkappa_{11} \varkappa_{12} \ldots . \varkappa_{1\nu_1} \mid \varkappa_{21} \varkappa_{22} \ldots . \varkappa_{2\nu_2} \mid \varkappa_{31} \varkappa_{32} \ldots . \varkappa_{3\nu_3} \mid \ldots .).$$

In ihr bedeutet der Klammerausdruck das Product$(\nu_1 - 1)!(\nu_2 - 1)!(\nu_3 - 1)!$ .... und die τ sowohl die Anzahl gleicher Zahlen in Combinationen $\varkappa_{\mu_1} \varkappa_{\mu_2} \ldots .$ als auch die Anzahl von Gleichen Combinationen. Die Summe erstreckt sich ueber alle Combinationen der Zahlen $\varkappa_1 \varkappa_2 \ldots . \varkappa_\nu$ bei denen die $\varkappa_{\mu_1} + \varkappa_{\mu_2} - \varkappa_{\mu_3} \ldots . = \lambda_\mu$ sind. In der Formel

$$\beta_{\kappa_1} \beta_{\kappa_2} \ldots . \beta_{\kappa_\nu} = \Sigma (-1)^{\nu + \nu_1 + \rho_1 + \rho_2 + \ldots\ldots} \varkappa_1 \varkappa_2 \ldots \mid \mu_1 \mu_2 \ldots . \, a_{\mu_1} a_{\mu_2} \ldots . a_{\mu_{\nu_1}}$$

lassen sich die Coefficienten $\varkappa_1 \varkappa_2 \ldots . \mid \mu_1 \mu_2 \ldots .$ mittelst der Formel berechnen

$$\varkappa_1 \varkappa_2 \ldots . \mid \mu_1 \mu_2 \ldots . = \Sigma g_\lambda \frac{1}{\tau_1 ! \, \tau_2 !} (\varkappa_{11} \varkappa_{12} \ldots . \mid \varkappa_{21} \varkappa_{22} \ldots . \mid \ldots .)$$

$$(\mu_{11} \mu_{12} \ldots . \mid \mu_{21} \mu_{22} \ldots . \mid \ldots .)$$

wo die $\kappa_{\sigma_1} \kappa_{\sigma_2} \ldots . \mu_{\sigma_1} \mu_{\sigma_2} \ldots .$ Combinationen der $\varkappa$ und μ darstellen, welche durch die Relationen ver knuepft sind $\varkappa_{\sigma_1} + \varkappa_{\sigma_2} + \ldots . = \mu_{\sigma_1} + \mu_{\sigma_2} + \ldots . = \lambda_\sigma$.

Sie sehen in beiden Fällen geschieht die Berechnung durch independte Formeln und nicht durch Recursionen. Mit eineger Uebung genuegen bis zum Grade 8 diese Formeln zur Berechnung eines Coefficienten ca 3 Minuten, Recursionsformeln sind nicht mehr noethig."

It was not until November, 1899, that opportunity was found to look over the letter to M. Hermite (Comptes Rendus 127, 1898, page 539). On examination difficulties were found in it, which the writer in a letter, November 9, 1899, stated to Professor Gordan, together with a sketch of proposals for their removal. The privilege of publishing the letter of October 17, 1898, was also asked. In reply November 21, 1899, Professor Gordan wrote as follows: "Ihre Behandlung der Aufgabe ist richtig und stimmt im Wesentlichen mit derjenigen uebereín, welche ich in Muenchen vorgetragen habe. Meine Arbeit habe ich vor ca 6 Monaten an die Annalen gesandt, sie ist aber noch nicht gedruckt; sobald diess geschieht werde ich Ihnen ein Exemplar senden. Wenn sie selbst etwas darueber schreiben wollen, so wird mich das sehr freuen. Mine Darstellung in dem Briefe an Hermite war noch unreif; ich habe sie seither verbessert."

The writer has not yet received a copy of Professor Gordan's Munich paper, and what follows is substantially the elaboration of the sketch sent to Professor Gordan November 9, 1899. The working out of his results is given in much more detail than was needed in the sketch.

§ 2. THE FUNCTIONS f, ϕ, ψ.

The functions

$$f=x^m+a_1x^{m-1}+\dots\dots+a_m\dots\dots(1)$$

$$\phi=x^n+b_1x^{n-1}+\dots\dots+b_n\dots\dots(2)$$

$$\psi=1+b_1x+\dots\dots+b_mx^n\dots\dots(3)$$

are three integral functions of the mth and nth degrees, respectively. Let the roots of f be α_1, α_2, ...α_m, and those of ϕ, β_1, β_2, ...β_n, then the roots of ψ are $1/\beta_1$, $1/\beta_2$, ...$1/\beta_n$, *i. e.*, the roots of ψ are the reciprocals of roots of ϕ. We have:

$$\psi=(1-\beta_1x)(1-\beta_2x)\dots\dots(1-\beta_nx)\dots\dots(4).$$

§ 3. THE EXPONENTIAL FORM OF ψ.

By developing $\log\psi=\sum_{i=1}^{i=n}\log(1-\beta_ix)\dots\dots(5)$ which we may always certainly do, so long as we take x to satisfy the inequality $\text{mod}(\beta x)<1$, where β denotes that root for which $\text{mod}(\beta)\geq\text{mod}(\beta_i)$, we get

$$\log\psi=-\sum_{r=1}^{r=\infty}(1/r)s_r(b)x^r\dots\dots(6)$$

where $s_r(b)$ denotes the sum of the rth powers of the β_i's, or we may write

$$\psi(x)=e^{-\sum_{r=1}^{r=\infty}(1/r)s_r(b)x^r}\dots\dots(7)$$

for values of x satisfying the inequality $\text{mod}(x)<\text{mod}(1/\beta)$, and this is the exponential form of ψ.

§ 4. THE RESULTANT OF f AND ψ.

If we choose those functions f, whose roots satisfy the inequality $\operatorname{mod}(\alpha) \leqq \operatorname{mod}(x) < \operatorname{mod}(1/\beta)$, where α is that root for which $\operatorname{mod}(\alpha) \geqq \operatorname{mod}(\alpha_i)$, then $\psi(\alpha_i) = e^{-\sum\limits_{r=1}^{r=\infty} (1/r) s_r(b) \alpha_i^r}$, and the resultant of f and ψ, $\prod\limits_{i=1}^{i=m} \psi(\alpha_i)$, becomes

$$R = e^{-\sum\limits_{r=1}^{r=\infty} (1/r) s_r(a) s_r(b)} \; \ldots\ldots (8),$$

where $s_r(a)$ denotes the sum of the rth powers of the α_i's. The formula (8) is *the exponential form* of the resultant of f and ψ, and it has been shown that it is true for such functions f and ψ which are so related that the inequality $\operatorname{mod}(\alpha) < \operatorname{mod}(1/\beta)$ is satisfied.

§ 5. THE EXPANSION OF R.

If we expand the exponential form given in the last section we obtain :

$$R = \sum_{\nu=0}^{\nu=\infty} \frac{(-1)^\nu}{\nu!} \left(\sum_{\lambda=1}^{\lambda=\infty} (1/\lambda) s_\lambda(a) s_\lambda(b) \right)^\nu \ldots\ldots (9).$$

By the multinomial theorem,

$$\left(\sum_{\lambda=1}^{\lambda=\infty} (1/\lambda) s_\lambda(a) s_\lambda(b) \right)^\nu = \sum \frac{\nu!}{\lambda_1^{\rho_1} \lambda_2^{\rho_2} \ldots\ldots \rho_1! \, \rho_2! \ldots\ldots} s_{\lambda_1}^{\rho_1}(a) s_{\lambda_1}^{\rho_1}(b) s_{\lambda_2}^{\rho_2}(a) s_{\lambda_2}^{\rho_2}(b) \ldots\ldots$$

where $\rho_1 + \rho_2 + \ldots\ldots = \nu$, hence,

$$R = \sum_{\nu=0}^{\nu=\infty} \sum \frac{(-1)^{\rho_1 + \rho_2 + \ldots\ldots}}{\lambda_1^{\rho_1} \lambda_2^{\rho_2} \ldots\ldots \rho_1! \, \rho_2! \ldots\ldots} (s_{\lambda_1}(a) s_{\lambda_1}(b))^{\rho_1} (s_{\lambda_2}(a) s_{\lambda_2}(b))^{\rho_2} \ldots\ldots (10).$$

We shall call this *the transcendental form* of the resultant. And we here add this remark : Though the previous developments upon which (10) depends have not been shown to be true for all functions f and ψ, yet when the left member R and the symmetric functions $s_r(a)$, $s_r(b)$ are replaced, for admissible functions, by their values in terms of the a's and b's, the relation (10) is seen to be an identity in terms of the latter, and therefore true for all functions f and ψ.

§ 6. COMPLETE GENERALITY OF THE TRANSCENDENTAL FORM.

From the foregoing sections it will be observed that the transcendental form of the resultant is precisely the same in terms of the s's for every binary resultant whatever. In obtaining the transcendental expression for R, we stated it is true with forms f and ψ of the mth and nth degrees, but we should have come precisely to the same transcendental form in terms of the s's, if we had used functions of any other degrees. This is the reason why we omitted to affect the symbol R with the subscripts m and n. *The transcendental form expresses* *all binary resultants formally in the same form, and is a universal formula comprehending them all. It is the most general formula for binary resultants.*

§7. THE FINITE FORM OF THE RESULTANT.

As the resultant $R_{m,n}$ of two forms of the mth and nth degrees of the form f and ψ, must contain terms of weight mn in the a's, but no terms of higher weight, it follows that the terms of the transcendental form of weight greater than mn, taken together are, by the remark of §5, identically equal to zero, when the s's are replaced by their values in terms of the a's and b's, though this will not appear explicitly or formally in terms of the s's,* and for the finite form we have,

$$R_{m,n}=\Sigma\Sigma\frac{(-1)^{\rho_1+\rho_2+\dots\dots}}{\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\dots\dots\rho_1!\,\rho_2!\dots\dots}(s_{\lambda_1}(a)s_{\lambda_1}(b))^{\rho_1}(s_{\lambda_2}(a)s_{\lambda_2}(b))^{\rho_2}\dots\dots\dots(11)$$

where the summation is to be extended over all values of the λ's and ρ's for which $\lambda_1\rho_1+\lambda_2\rho_2+\dots\dots\leqq mn$.

Example: Let

$$f=x^2+a_1x+a_2$$
$$\phi=x+b_1$$
$$\psi=1+b_1x.$$

By the preceding formula or directly by elementary methods we find

$$R_{f,\,\psi}=R_{2,\,1}=1-s_1(a)s_1(b)+\tfrac{1}{2}(s_1(a)s_1(b))^2-\tfrac{1}{2}s_2(a)s_2)b).$$

The resultant of any two binary forms may be expressed in this manner by (11), for it is only necessary to supply for the $s(b)$'s those s's which belong to the form whose roots are the reciprocals of those of the given second form.

§8. GENERALITY OF THE FORMULA (11).

The form (11) which we may write, by using Gordan's abbreviations,

$$R_{m,\,n}=\Sigma\frac{(-1)^{\rho_1+\rho_2+\dots\dots\dots}}{\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\dots\dots\rho_1!\,\rho_2!\dots\dots}S_\lambda\,(a)\,S_\lambda\,(b)\quad\dots\dots(12)$$

with $\lambda_1\rho_1+\lambda_2\rho_2+\dots\dots\leqq mn$, represents formally all resultants $R_{\mu,\,\nu}$, for which $\mu\nu=mn$, in terms of the s's in precisely the same form; therefore it is still a very general and elastic form. In order to make it definite, one substitutes for the s's their values in terms of the a's and b's for definite equations whose degrees m_1, n_1, satisfy the condition $m_1n_1=mn$. Then it represents a definite resultant R_{m_1,n_1} of the given kind.

Example: $R_{12,1}$, $R_{1,12}$, $R_{3,\,4}$, $R_{4,\,3}$, $R_{2,\,6}$, and $R_{6,\,2}$, all have formally the same expression in terms of the s's.

The expression for $R_{m,n}$ will represent something which is not at all a resultant (something more or less than a resultant) if values of the s's in terms of the a's and b's are substituted which belong to equations where $m_1 n_1$ not equal to mn. In consequence of this the symbol $R_{m,n}$ includes not only resultants but also other expressions not resultants. We use the symbol in the following section in the enlarged sense.

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*And as Professor White has suggested, it is also well to state, what is involved in this, that all terms of the same weight beyond those of weight mn, are seen to vanish identically by themselves, when the s's are replaced by a's and b's.

§ 9. PARTIAL DIFFERENTIAL EQUATION WITH RESPECT TO THE s'S.

If we differentiate the form

$$R_{m,n}=\Sigma\frac{(-1)^{\rho_1+\rho_2+\cdots\cdots}}{\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\cdots\cdot\rho_1!\,\rho_2!\cdots\cdot}(s_{\lambda_1}(a)s_{\lambda_1}(b))^{\rho_1}(s_{\lambda_2}(a)s_{\lambda_2}(b))^{\rho_2}\cdots\cdots$$

of (11) partially with respect to $s_{\lambda_r}(a)$, we obtain

$$\lambda_r\frac{\partial R_{m,n}}{\partial s_{\lambda_r}(a)}=(-1)s_{\lambda_r}(b)\,\Sigma\frac{(-1)^{\rho_1+\rho_2+\cdots\cdots(\rho_r-1)+\cdots\cdot}}{\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\cdots\cdot\lambda_r{}^{\rho_r-1}\cdots\cdot\rho_1!\,\rho_2!\cdots\cdot(\rho_r-1)!\cdots\cdot}$$

$$(s_{\lambda_1}(a)s_{\lambda_1}(b))^{\rho_1}\cdots\cdot(s_{\lambda_r}(a)s_{\lambda_r}(b))^{\rho_r-1}\cdots\cdots \quad (13)$$

with $\lambda_1\rho_1+\lambda_2\rho_2+\cdots\cdot\leqq mn-\lambda_r$, and hence our equation is

$$\lambda_r\frac{\partial R_{m,n}}{\partial s_{\lambda_r}(a)}=(-1)s_{\lambda_r}(b)R_{m',n'},\text{ with } m'n'=mn-\lambda_r.$$

By repeating this process until we have differentiated ρ_r times we get

$$\frac{\lambda_r{}^{\rho_r}\partial^{\rho_r}R_{m,n}}{\partial s_{\lambda_r}{}^{\rho_r}(a)}=(-1)^{\rho_r}s_{\lambda_r}{}^{\rho_r}(b)R_{m',n'}\cdots\cdots(14)$$

where m', $n'=mn-\lambda_r\rho_r$, and in a similar way we obtain more generally still,

$$(-1)^{\rho_1+\rho_2+\cdots\cdots}\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\cdots\cdot\frac{\partial^{\rho_1+\rho_2+\cdots\cdots}R_{m,n}}{\partial s_{\lambda_1}{}^{\rho_1}\partial s_{\lambda_2}{}^{\rho_2}\cdots\cdots}=s_{\lambda_1}{}^{\rho_1}s_{\lambda_2}{}^{\rho_2}\cdots\cdot R_{m',n'}\cdots\cdots \quad (15).$$

$$m'n'=mn-(\lambda_1\rho_1+\lambda_2\rho_2+\cdots\cdot)\cdots\cdots(16)$$

where the differentiation may be distributed between the $s_\lambda(a)$'s and the $s_\lambda(b)$'s, and $R_{m',n'}$ may under certain circumstances be a resultant.

Examples: 1. $\dfrac{\partial R_{3,2}}{\partial s_2(a)}=-\frac{1}{2}s_2(b)R_{r,r'}$, where $rr'=4$, but $R_{r,r'}$ is neither the resultant $R_{4,1}$, nor $R_{2,2}$; the presence of terms $s_\lambda(a)$ where $\lambda>1$, which must contain b_2 prevents the former; the presence of terms $s_\lambda(a)$ where $\lambda>2$, which must contain a_3 prevents the latter,

2. $\dfrac{\partial R_{6,1}}{\partial s_5(a)}=-\frac{1}{5}s_5(b)R_{r,r'}$, where $rr'=1$, and here $R_{r,r'}=R_{1,1}$, is the resultant of $x+a_1$, and $1+b_1x$.

§ 10. DIFFERENTIATION OF THE TRANSCENDENTAL FORM.

The same differentiation applied to the transcendental form (10), or what is the same thing (as is seen by performing the differentiation in both cases) to the exponential form (8) is questionable. By (8) we have $R = e^{-\Sigma(1/r)s_r(a)s_r(b)}$. Differentiating with respect to $s_\lambda(a)$,

$$\frac{\lambda \partial R}{\partial s_\lambda(a)} = (-1)e^{-\Sigma(1/r)s_r(a)s_r(b)}s_\lambda(b), \text{ or } \lambda\frac{\partial R}{\partial s_\lambda(a)} = (-1)s_\lambda(b)R \;\ldots\ldots(17),$$

with a corresponding formula

$$(-1)^{\rho_1+\rho_2+\cdots\cdots}\lambda_1^{\rho_1}\lambda_2^{\rho_2}\ldots\ldots\frac{\partial^{\rho_1+\rho_2+\cdots\cdots}R}{\partial s_{\lambda_1}^{\rho_1}\partial s_{\lambda_2}^{\rho_1}\ldots\ldots} = s_{\lambda_1}^{\rho_1}s_{\lambda_2}^{\rho_2}\ldots\ldots R\ldots\ldots(18)$$

for the repitition. If (18) is allowed to stand it must be with the understanding that it is the universal type of all differential equations of the kind, that the R involved is the transcendental form, and that in any particular case such values in finite form must be picked out from the infinity of values which R admits, as will throw the indefinite form of (18) into that of (15) by which (18), if at all allowed, must be interpreted and explained.

II. APPLICATION TO SYMMETRIC FUNCTIONS.

§ 11. THE GENERAL TERM IN THE EXPANSION OF $\Sigma\alpha_1^{\kappa_1}\alpha_2^{\kappa_2}\ldots\ldots\alpha_\nu^{\kappa_\nu}$.

1. *In Terms of the s's.*

In terms of symmetric functions, the resultant $R_{m,n}$ of f and ϕ consists of terms of the form $b_{\kappa_1}b_{\kappa_2}\ldots\ldots b_{\kappa_\nu}\Sigma\alpha_1^{\kappa_1}\alpha_2^{\kappa_2}\ldots\ldots\alpha_\nu^{\kappa_\nu}$. On the other hand we may express $R_{m,n}$ by (11) as a sum of terms of the form

$$\frac{(-1)^{\rho_1+\rho_2\cdots\cdots\rho_\kappa}}{\lambda_1\lambda_2\ldots\ldots\lambda_\mu\rho_1!\,\rho_2!\ldots\ldots\rho_\kappa!}s_{\lambda_1}(a)s_{\lambda_2}(a)\ldots\ldots s_{\lambda_\mu}(a)s_{\lambda_1}(b)s_{\lambda_2}(b)\ldots\ldots s_{\lambda_\mu}(b)\ldots\ldots(19)$$

by slightly changing the notation, ρ_1 being the number of λ's equal to λ_1, ρ_2 the number equal to λ_2, etc, and the summation to extend over all values of the λ's for which $\lambda_1+\lambda_2+\ldots\ldots\lambda_\mu \leqq mn$. In the term of $R_{m,n}$, as so expressed, we pick out the portion containing $b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\nu}$ from the product $s_{\lambda_1}(b)s_{\lambda_2}(b)\ldots\ldots s_{\lambda_\mu}(b)$, and obtain

$$(-1)^{\nu+\rho_1-\rho_2+\cdots\cdots}C_{\lambda,\kappa}b_{\kappa_1}b_{\kappa_2}\ldots\ldots b_{\kappa_\mu}s_{\lambda_1}(a)s_{\lambda_2}(a)\ldots\ldots s_{\lambda_\mu}(a)$$

where $\varkappa_1+\varkappa_2+\ldots\ldots\varkappa_\nu = \lambda_1+\lambda_2+\ldots\ldots\lambda_\mu$. By Waring's formula,

$$s_{\lambda_r}(b) = \lambda_r\Sigma(-1)^{\tau_1+\tau_2+\ldots\ldots\tau_n}\frac{(\tau_1+\tau_2+\ldots\ldots\tau_n-1)!}{\tau_1!\,\tau_2!\ldots\ldots\tau_n!}b_1^{\tau_1}\,b_2^{\tau_2}\ldots\ldots b_n^{\tau_n}$$

with $\tau_1+2\tau_2+\ldots\ldots n\tau_n = \lambda_r$. Among the terms of $s_{\lambda_r}(b)$ is

$$\frac{\lambda_r(-1)^{\nu_r}(\nu_r-1)!}{\tau_1!\tau_2!\ldots\ldots}b_{\kappa_{r1}}\,b_{\kappa_{r2}}\ldots\ldots b_{\kappa_{r\nu}}$$

where $\varkappa_{r1}$, $\varkappa_{r2}\ldots\ldots$ form a set of numbers found among $\varkappa_1$, $\varkappa_2$, $\ldots\ldots\varkappa_\nu$, and τ_1

of the set are equal to $\varkappa_{r_1}$, τ_2 to $\varkappa_{r_2}$, etc., and where $\varkappa_{r_1}+\varkappa_{r_2}+\ldots\ldots\varkappa_{r_{\nu_r}}=\lambda_r$. In the same way from from $r=1$, to $r=\mu$, we obtain from μ sets with $\gamma_1+\gamma_2+\ldots\ldots\gamma_\mu=\nu$, similar factors, and therefore a term containing $b_{\kappa_1}b_{\kappa_2}\ldots\ldots b_{\kappa_\nu}$ in (19) is

$$\frac{(-1)^{\rho_1+\rho_2+\ldots\ldots\nu}}{\rho_1!\,\rho_2!\ldots\ldots\tau_1!\,\tau_2!\ldots\ldots}(\nu_1-1!)(\nu_2-1)!\ldots(\nu_\mu-1)!s_{\lambda_1}(a)\ldots s_{\lambda_\mu}(a)b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\nu}\ldots(20)$$

In like manner other decompositions, if there be such, of the $\nu\varkappa$'s into μ sets, where the sum of the sets is equal to λ_1, $\lambda_2\ldots\ldots\lambda_\mu$ respectively, will give similar expressions, and the entire coefficient of $b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\nu}s_{\lambda_1}(a)\ldots s_{\lambda_\mu}(a)$ will be

$$\Sigma\frac{(-1)^{\rho_1+\rho_2+\ldots\ldots\rho_\kappa+\nu}}{\rho_1!\rho_2!\ldots\rho_\kappa!\,\tau_1!\tau_2!\tau_r!\ldots}(\nu_1-1)!(\nu_2-1)!\ldots(\nu_\mu-1)!=(-1)^{\rho_1+\rho_2+\ldots\ldots\nu}C_{\lambda,\,\kappa}\ldots(21).$$

Another term $s_{\mu_1}(a)s_{\mu_2}(a)\ldots\ldots s_{\mu_\kappa}(a)b_{\kappa_1}b_{\kappa_2}\ldots\ldots b_{\kappa_\nu}$ will have a coefficient similarly formed, etc. Equating the terms in both developments of $R_{m,n}$ which contain $b_{\kappa_1}b_{\kappa_2}\ldots\ldots b_{\kappa_\nu}$ and dividing out this product, we have,

$$\Sigma\alpha_1^{\kappa_1}\alpha_2^{\kappa_2}\ldots\ldots\alpha_\nu^{\kappa_\nu}=\Sigma(-1)^{\nu+\rho_1+\rho_2+\ldots\ldots\rho_\kappa}C_{\lambda,\,\kappa}S_\lambda(a)\ldots\ldots(22).$$

It will be seen that this formula agrees with that given in Faà di Bruno's Binäre Formen, page 8, except that the terms are grouped differently in the summation, formula (22) grouping the terms with respect to the numerical values of the subscripts of the s's, the latter grouping them with respect to the number of elements entering into a combination to form the subscript. The coefficient $(-1)^{\nu+\rho_1+\rho_2+\ldots\ldots}C_{\lambda,\,\kappa}$ of (21) is the coefficient of the general term $S_\lambda(a)$ in the expansion of $\Sigma\alpha_1^{\kappa_1}\alpha_2^{\kappa_2}\ldots\ldots\alpha_\nu^{\kappa_\nu}$ in terms of the s's.

2. *In Terms of the a's.*

By using the preceding formula (22), we easily obtain a formula for the expansion of $\Sigma\alpha_1^{\kappa_1}\alpha_2^{\kappa_2}\ldots\ldots\alpha_\nu^{\kappa_\nu}$ in terms of the a's. We have

$$s_{\lambda_r}(a)=\lambda_r\Sigma(-1)^{\tau_1+\tau_2+\ldots\ldots\tau_m}\frac{(\tau_1+\tau_2+\ldots\ldots\tau_m-1)!}{\tau_1!\,\tau_2!\ldots\ldots\tau_m!}a_1^{\tau_1}a_2^{\tau_2}\ldots\ldots a_m^{\tau_m}$$

with $\tau_1+2\tau_2+\ldots\ldots m\tau_m=\lambda_r$. Among the terms of $s_{\lambda_r}(a)$ is

$$\frac{(-1)^{\sigma_r}\lambda_r(\sigma_r-1)!}{\varkappa_1!\varkappa_2!\ldots\ldots\varkappa_\rho!}a_{\mu_{r_1}}a_{\mu_{r_2}}\ldots\ldots a_{\mu_{r_{\sigma_r}}}$$

where $\mu_{r_1}+\mu_{r_2}+\ldots\mu_{r_{\sigma_r}}=\lambda_r$, the $\varkappa$'s are for repetitions in the a's, $\sigma_1+\sigma_2+\ldots\sigma_\mu=\sigma$, and μ_{r_1}, $\mu_{r_2}\ldots\ldots$ are found among μ_1, μ_2, $\ldots\ldots\mu_\sigma$; similarly for other factors. Then a term in $(-1)^{\nu+\rho_1+\rho_2+\ldots\ldots}C_{\lambda,\,\kappa}S_\lambda(a)$ containing $a_{\mu_1}a_{\mu_2}\ldots\ldots a_{\mu_\sigma}$ is

$$(-1)^{\sigma+\nu+\rho_1+\rho_2+\ldots\ldots}$$

$$\frac{\lambda_1\lambda_2\ldots\ldots}{\varkappa_1!\varkappa_2!\ldots\ldots}C_{\lambda,\,\kappa}(\sigma_1-1)!(\sigma_2-1)!\ldots\ldots(\sigma_\mu-1)!a_{\mu_1}a_{\mu_2}\ldots\ldots a_{\mu_\sigma}\ldots\ldots(23).$$

The sum of all numerical coefficients of the form contained in (23) is the complete coefficient of $a_{\mu_1}a_{\mu_2}\ldots\ldots a_{\mu_\sigma}$ in the expansion of $\Sigma\alpha_1^{\kappa_1}\alpha_2^{\kappa_2}\ldots\ldots\alpha_\nu^{\kappa_\nu}$ and this in his notation Gordan expresses as

$$\Sigma(-1)^{\sigma+\nu+\rho_1+\rho_2+\ldots\ldots}\frac{g_\lambda}{\tau_1!\,\tau_2!\ldots\ldots}(\varkappa_{11}\varkappa_{12}\ldots\ldots\mid\varkappa_{21}\varkappa_{22}\ldots\ldots\mid\ldots\ldots)$$

$$(\mu_{11}\mu_{12}\ldots\ldots\mid\mu_{21}\mu_{22}\ldots\ldots\mid\ldots\ldots)\ldots\ldots(24).$$

By means of this formula the general coefficient in the expansion of the binary resultant in terms of the a's and b's can of course be independently expressed.

Elmira, New York, December 19, 1899.